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Khovanov's Heisenberg category, moments in free probability, and shifted symmetric functions

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Abstract. We establish an isomorphism between the center $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ of Khovanov's Heisenberg category $\mathcal{H}'$ and the algebra Λ^* of shifted symmetric functions defined by Okounkov-Olshanski. We give a graphical description of the shifted power and Schur bases of Λ^* as elements of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$, and describe the curl generators of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ in the language of shifted symmetric functions. This latter description makes use of the transition and co-transition measures of Kerov and the noncommutative probability spaces of Biane.

Keywords: shifted symmetric functions, Heisenberg algebra categorification, asymptotic representation theory of symmetric groups, noncommutative probability theory

1 Introduction

In [10], Khovanov introduces a graphical calculus of oriented planar diagrams and uses it to define a linear monoidal category $\mathcal{H}'$, which he proposes as a categorification of the Heisenberg algebra. We denote by $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ the endomorphism algebra of the monoidal unit in $\mathcal{H}'$. The commutative algebra $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ is, by definition, the algebra of closed oriented planar diagrams modulo the relations of the Khovanov graphical calculus. In his study of morphism spaces of $\mathcal{H}'$, Khovanov introduces two sets of generators for $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$: the clockwise curls $\{c_k\}_{k\geqslant 0}$ and the counterclockwise curls $\{\tilde{c}_k\}_{k\geqslant 2}$. He then establishes algebra isomorphisms

$$\mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \cong \mathbb{C}[c_0, c_1, c_2, \dots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \dots],$$

and describes a recursion for expressing the clockwise and counterclockwise curls in terms of each other. He then relates $\mathcal{H}'$ to representation theory by defining a sequence of monoidal functors $f_k^{\mathcal{H}'}$ from $\mathcal{H}'$ to bimodule categories for symmetric groups. A consequence of the existence of these functors is the existence of surjective algebra homomorphisms,

$$f_n^{\mathcal{H}'} : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow Z(\mathbb{C}[S_n]),$$

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from $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ to the center of the group algebra of each symmetric group. Based in part on this, Khovanov suggests that there should be a close connection between $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and the asymptotic representation theory of symmetric groups. Furthermore, one might hope that $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ in fact gives a diagrammatic description of some algebra of pre-existing combinatorial interest.

The main goal of the current paper is to make precise the connection between $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and both the asymptotic representation theory of symmetric groups and algebraic combinatorics. We do this by establishing an isomorphism between

$$\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow \Lambda^*,$$

where Λ^* is the *shifted symmetric functions* of Okounkov-Olshanski [13]. (See Theorem 11.) The algebra of shifted symmetric functions Λ^* is a deformation of the algebra of symmetric functions. As is the case for $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$, there are surjective algebra homomorphisms

$$f_n^{\Lambda^*} : \Lambda^* \longrightarrow Z(\mathbb{C}[S_n]),$$

to the center of the group algebra of each symmetric group. The isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow \Lambda^*$ is canonical, in that it intertwines the homomorphisms $f_n^{\mathcal{H}'}$ and $f_n^{\Lambda^*}$.

The isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow \Lambda^*$ allows us to give a graphical description of several important bases of Λ^*. For example, the shifted power sum denoted $p^{\#}_{\lambda}$ in [13] appears in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ as the closure of a permutation of cycle type λ. The shifted Schur function s^*_{λ} appears as the closure of a Young symmetrizer of type λ. (See Corollary 12).

In the other direction, it is also reasonable to ask for a description of the image of Khovanov's curl generators c_k and $\tilde{c}_k$ as elements of Λ^*. It turns out that the right language for such a description is that of noncommutative probability theory. In [7], Kerov introduces, for each partition λ, a pair of finitely supported probability measures on $\mathbb{R}$; these probability measures are known as the *transition* and *co-transition* measures, or sometimes as growth and decay. In work of Biane [1], these probability measures appear as the compactly-supported measures associated to self-adjoint operators on a noncommutative probability space, and as a result they are basic objects of interest at the intersection of representation theory and noncommutative probability theory. In particular, the *moments* and *Boolean cumulants* of the transition and co-transition measures may be regarded as elements of Λ^*. In Theorem 13, we show that the isomorphism φ takes Khovanov's curl generators c_k and $\tilde{c}_k$ to scalar multiples of the kth moments of Kerov's transition and co-transition measures. In fact, the close relationship between the transition and co-transition measures themselves yields two independent descriptions of the image of the curl generator c_k: it is equal to a scalar multiple of both the kth moment of the co-transition measure and the $(k+2)$th Boolean cumulant of the transition measure. The observation that the Boolean cumulants of the transition measure are equal to the moments of the co-transition measure seems to be new, and is closely connected to the adjointness of induction and restriction functors between representation categories of symmetric groups. A dictionary between several of the bases of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and Λ^* is given in Table 1.

The existence of a relationship between $\mathcal{H}'$ and free probability – and indeed, much of this paper – was anticipated by Khovanov in [10]. The relationship between generators of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and the noncommutative probability spaces of [1] may be seen as a further manifestation of the "planar structure" of free probability; the many connections between noncommutative probability and other mathematical subjects with planar structure are emphasized in the work of Guionnet, Jones and Shlyakhtenko [3].

This text is an extended abstract of the preprint [11], where complete proofs and additional background can be found.

2 The symmetric group and its normalized character theory

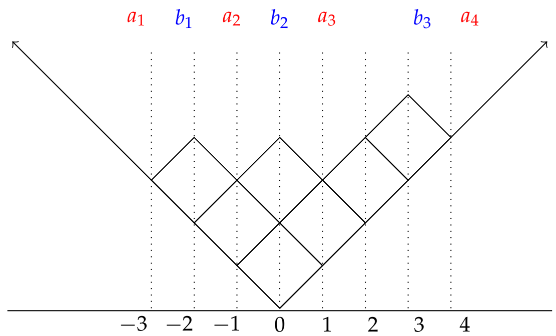
We begin by establishing notation related to partitions and Young diagrams. Let $\mathcal{P}_n$ be the set of partitions of n and $\mathcal{P} := \bigcup_{n\geqslant 0}\mathcal{P}_n$. We freely identify $\mu \in \mathcal{P}$ with its corresponding Young diagram, which we draw using Russian notation (see Example 1). If $\square$ is a cell in the ith row and jth column of μ then the *content* of $\square$ is defined as $\mathrm{cont}(\square) := j - i$. We say that a cell $\square \notin \mu$ is *i-addable* with respect to μ if it has content i and adding it to μ gives a Young diagram. We say that a cell $\square \in \mu$ is *i-removable* with respect to μ if it has content i and removing it from μ gives a Young diagram. We call two sequences $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ *interlacing* when

$$a_1 < b_1 < a_2 < \cdots < a_{d-1} < b_{d-1} < a_d.$$

The *center* of this pair of sequences is defined as the quantity $(a_1 + \cdots + a_d) - (b_1 + \cdots + b_{d-1})$. There is a bijection between Young diagrams and pairs of integer-valued interlacing sequences $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ with center 0. Given μ the corresponding sequence $a_1, \ldots, a_d$ is the ordered list of all a_j such that there exists an a_j-addable cell with respect to μ, while $b_1, \ldots, b_{d-1}$ is the ordered list of all b_j such that there exists a b_j-removable cell with respect to μ. It is clear that $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ are interlacing. We denote by $\mu^{(j)}$ the Young diagram that we obtain by adding a cell of content a_j, so that $\mathrm{cont}(\mu^{(j)}/\mu) = a_j$. Similarly, we denote by $\mu_{(j)}$ the Young diagram that we obtain by removing a cell of content b_j from μ, so that $\mathrm{cont}(\mu/\mu_{(j)}) = b_j$.

Example 1. *Let $\mu = (4, 2, 1)$. Then μ yields the interlacing sequences*

$$-3 < -1 < 1 < 4 \qquad \textit{and} \qquad -2 < 0 < 3.$$



Let S_n be the symmetric group with Coxeter generators $s_1, \ldots, s_{n-1}$. If $g \in S_n$ has cycle type $\lambda \in \mathcal{P}_n$, then we write $\mathrm{sh}(g) := \lambda$. For $k \leqslant n$, there is an embedding $\iota_{k,n} : \mathbb{C}[S_k] \hookrightarrow \mathbb{C}[S_n]$ called the *standard embedding* which sends S_k to the subgroup generated by $s_1, \ldots, s_{k-1}$.

Let L^λ be the simple $\mathbb{C}[S_n]$-module (i.e. the irreducible S_n representation) associated to $\lambda \in \mathcal{P}_n$ and $\chi^\lambda : \mathbb{C}[S_n] \to \mathbb{C}$ its character. Abusing notation, we write $\chi^\lambda(\mu)$ for $\chi^\lambda(g)$ when $\mathrm{sh}(g) = \mu$. The *normalized character* $\widetilde{\chi}^\lambda : \bigoplus_{k \leqslant n} \mathbb{C}[S_k] \to \mathbb{C}$ associated to λ is defined so that for $x \in \mathbb{C}[S_k]$,

$$\widetilde{\chi}^\lambda(x) := \frac{\chi^\lambda(\iota_{k,n}(x))}{\dim L^\lambda} = \frac{\chi^\lambda(\iota_{k,n}(x))}{\chi^\lambda(1)}. \tag{2.1}$$

Definition 2. *For $\mu = (\mu_1, \ldots, \mu_t) \in \mathcal{P}_k$ with $k \leqslant n$, set*

$$A_{\mu,n} = \sum (i_1, \ldots, i_{\mu_1}) \ldots (i_{k-\mu_t+1}, \ldots, i_k) \tag{2.2}$$

where this sum is taken over all distinct k-tuples $(i_1, \ldots, i_k)$ of elements from $\{1, 2, \ldots, n\}$. We call $A_{\mu,n}$ the normalized conjugacy class sum *associated to μ in $\mathbb{C}[S_n]$.*

The elements $A_{\mu,n}$ belong to $Z(\mathbb{C}[S_n])$ and for $\lambda \in \mathcal{P}_n$

$$\widetilde{\chi}^\lambda(A_{\mu,n}) = (n \downharpoonleft k) \frac{\chi^\lambda(\mu \cup 1^{n-k})}{\dim L^\lambda} \tag{2.3}$$

where $(n \downharpoonleft k)$ is the *falling factorial power*, which is defined as $(n \downharpoonleft k) = n(n-1)\ldots(n-k+1)$ for integers k, n with $0 < k \leqslant n$.

Finally, recall that the Jucys-Murphy elements $\{J_i\}_{1 \leqslant k \leqslant n} \subseteq \mathbb{C}[S_n]$, are defined as

$$J_1 = 0, \qquad \text{and} \qquad J_k = (1,k) + (2,k) + \cdots + (k-1,k), \qquad 2 \leqslant k \leqslant n.$$

2.1 The transition measure and co-transition measure

In this section we review the transition and co-transition measures associated to a Young diagram. Assume that $\lambda \in \mathcal{P}_n$ and let $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ be the interlacing sequences associated to λ. Recall that $\lambda^{(1)}, \ldots, \lambda^{(d)}$ are the partitions of $n+1$ such that

$\mathrm{cont}(\lambda^{(i)}/\lambda) = a_i$, while $\lambda_{(1)}, \ldots, \lambda_{(d-1)}$ are the partitions of $n-1$ such that $\mathrm{cont}(\lambda/\lambda_{(i)}) = b_i$.

For λ, the *transition measure* $\widehat{\omega}_\lambda$ and *co-transition measure* $\check{\omega}_\lambda$ on $\mathbb{R}$ are defined as

$$\widehat{\omega}_\lambda := \sum_{i=1}^{d} \frac{\dim(L^{\lambda^{(i)}})}{(n+1)\dim(L^\lambda)}\delta_{a_i} \quad \text{and} \quad \check{\omega}_\lambda := \sum_{i=1}^{d-1} \frac{\dim(L^{\lambda_{(i)}})}{\dim(L^\lambda)}\delta_{b_i}$$

respectively, where δ_x is the Dirac delta measure with support on $x \in \mathbb{R}$. These probability measures were first investigated by Kerov [7], [8]. They are fundamental tools in the study of the asymptotic representation theory of symmetric groups and its connection to free probability.

The kth moments associated to $\widehat{\omega}_\lambda$ and $\check{\omega}_\lambda$ are given by

$$\widehat{m}_k(\lambda) = \sum_{i=1}^{d} \frac{\dim(L^{\lambda^{(i)}})}{(n+1)\dim(L^\lambda)}a_i^k \quad \text{and} \quad \check{m}_k(\lambda) = \sum_{i=1}^{d-1} \frac{\dim(L^{\lambda_{(i)}})}{\dim(L^\lambda)}b_i^k$$

respectively. *Boolean cumulants* linearize convolution of probability measures under the notion of Boolean independence [14] and can be defined recursively such that if $\{\widehat{b}_k(\lambda)\}_{k\geqslant 1}$ are the Boolean cumulants associated to $\widehat{\omega}_\lambda$ then,

$$\sum_{i=1}^{k} \widehat{m}_{k-i}(\lambda)\widehat{b}_i(\lambda) = \widehat{m}_k(\lambda). \tag{2.4}$$

Proposition 3. *Let $\lambda \in \mathcal{P}$ and $k \geqslant 0$, then $\widehat{b}_1(\lambda) = 0$ and $\widehat{b}_{k+2}(\lambda) = |\lambda|\check{m}_k(\lambda)$.*

There is a more algebraic approach to the transition measure due to Biane [1]. Let $\mathrm{pr}_{n-1} : \mathbb{C}[S_n] \to \mathbb{C}[S_{n-1}] \subset \mathbb{C}[S_n]$ be the projection map so that for $g \in S_n$, $\mathrm{pr}_{n-1}(g) = g$ if $g \in S_{n-1}$ and 0 otherwise.

Proposition 4. *For $\lambda \in \mathcal{P}_n$,*

$$\widehat{m}_k(\lambda) = \widetilde{\chi}^\lambda[pr_n(J_{n+1}^k)] \tag{2.5}$$

and

$$\widehat{b}_{k+2}(\lambda) = |\lambda|\check{m}_k(\lambda) = \widetilde{\chi}^\lambda\Big(\sum_{i=1}^{n} s_i \ldots s_{n-1}J_n^k s_{n-1} \ldots s_i\Big). \tag{2.6}$$

Proof. The statement of (2.5) appears in [2, Section 4]. A detailed proof can be found in [4, Theorem 9.23]. (2.6) follows from the fact that $\widetilde{\chi}^\lambda$ is a class function and from the spectral decomposition of J_n [15]. □

Proposition 4 is related to the fact that we are working in a noncommutative probability space (that is, a von Neumann algebra equipped with a normal faithful trace). In our case the algebra is $\mathrm{End}(L^\lambda) \otimes M_{n+1}(\mathbb{C})$ and $\widehat{\omega}_\lambda$ then arises from the distribution of a self-adjoint element in this algebra [1, Proposition 3.3].

3 The shifted symmetric functions Λ^*

The *algebra of shifted symmetric functions* Λ^* is a deformation of the classical symmetric functions Λ. Elements of Λ^* are "shifted symmetric", that is, they become symmetric in the new variables $x_i' = x_i - i$. For a detailed study of Λ^*, see [13]. Λ^* contains shifted analogs of elements from Λ. These include the *shifted Schur functions* $\{s^*_\lambda\}_{\lambda\in\mathcal{P}}$ [13], as well as the *elementary shifted functions* $\{e^*_k\}_{k\geqslant 0}$ and *complete shifted functions* $\{h^*_k\}_{k\geqslant 0}$ defined by $e^*_k := s^*_{(1^k)}$ and $h^*_k := s^*_{(k)}$ respectively. Let F be the linear isomorphism $F : \Lambda \to \Lambda^*$ which sends the classical Schur function $s_\lambda \mapsto s^*_\lambda$. Define the element $p^\#_\lambda \in \Lambda^*$ to then be

$$p^\#_\lambda := F(p_\lambda), \tag{3.1}$$

where p_λ is the power sum symmetric function in Λ. The elements $p^\#_\lambda$ are one of several shifted analogues of the power sums. $p^\#_1,\ p^\#_2,\ p^\#_3 \dots$ are algebraically independent and generate Λ^* [6]. Note that unlike classical power sum symmetric functions, in general $p^\#_\lambda \neq p^\#_{\lambda_1} p^\#_{\lambda_2} \dots p^\#_{\lambda_r}$ for $\lambda = (\lambda_1, \dots, \lambda_r)$.

3.1 Λ^* as functions on $\mathcal{P}$

Let $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$ be the algebra of functions from $\mathcal{P}$ to $\mathbb{C}$ with pointwise multiplication. Viewing $\mu = (\mu_1, \dots, \mu_t) \in \mathcal{P}$ as the sequence $(\mu_1, \dots, \mu_t, 0, 0, \dots)$, we can evaluate $f \in \Lambda^*$ on μ by setting

$$f(\mu) = f(\mu_1, \dots, \mu_t, 0, 0, \dots). \tag{3.2}$$

Since $(\mu_1, \dots, \mu_t, 0, 0, \dots)$ has only a finite number of nonzero values, (3.2) is well-defined. In fact f is uniquely defined by its values on $\mathcal{P}$. Thus Λ^* may be realized as a subalgebra of $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$ [9], [13].

Proposition 5. *[13] For $\mu \in \mathcal{P}_k$, $\lambda \in \mathcal{P}_n$,*

$$p^\#_\mu(\lambda) = \begin{cases} \frac{(n\downarrow k)}{\dim L^\lambda} \chi^\lambda(\mu \cup 1^{n-k}) & k \leqslant n \\ 0 & \textit{otherwise.} \end{cases} \tag{3.3}$$

Remark 6. *We will later use the fact that $p^\#_1 = x_1 + x_2 + \dots$, so that $p^\#_1(\lambda) = |\lambda|$ for all $\lambda \in \mathcal{P}$.*

In Section 2.1 we introduced the moments $\{\widehat{m}_k(\lambda)\}$ (respectively $\{\check{m}_k(\lambda)\}$) of the transition measure (respectively co-transition measure) associated to a partition λ and the corresponding Boolean cumulants $\{\widehat{b}_k(\lambda)\}$. We can interpret all of these as elements of $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$ via

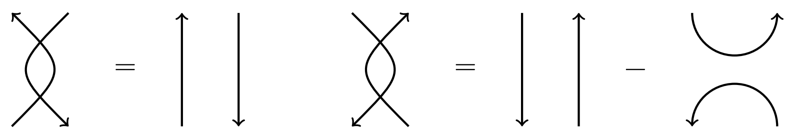
$$\lambda \overset{\widehat{m}_k}{\longmapsto} \widehat{m}_k(\lambda), \quad \lambda \overset{\check{m}_k}{\longmapsto} \check{m}_k(\lambda), \quad \text{and} \quad \lambda \overset{\widehat{b}_k}{\longmapsto} \widehat{b}_k(\lambda).$$

Proposition 7. *[12, Theorem 6.4] As elements of $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$, $\widehat{m}_k$ and $\widehat{b}_k$ belong to Λ^*.*

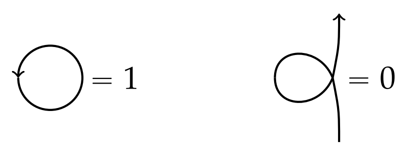
Remark 8. *In [12] Section 5, Lassalle shows that with the appropriate alphabet A_λ (which is specific to each partition λ), $\widehat{m}_k(\lambda) = h_k(A_\lambda)$ and $\widehat{b}_k(\lambda) = (-1)^{k-1} e_k(A_\lambda)$.*

4 The algebra $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$

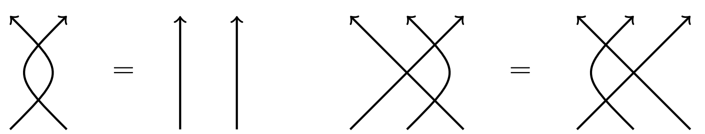
In [10], Khovanov defined an additive $\mathbb{C}$-linear monoidal category $\mathcal{H}'$ which we will call the *Heisenberg category*. The unit object in $\mathcal{H}'$ is denoted by $\mathbf{1}$. In this paper we study the endomorphism algebra $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$. $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ is a $\mathbb{C}$-algebra generated by planar diagrams modulo local relations. The diagrams are closed oriented compact 1-manifolds immersed in the strip $\mathbb{R} \times [0,1]$, modulo isotopy. Multiplication corresponds to juxtaposition of diagrams. The local relations are:



(4.1)



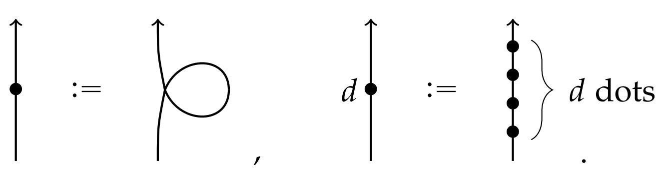
(4.2)



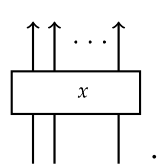
(4.3)

The relations (4.1)-(4.2) are motivated by the Heisenberg relation $pq = qp + 1$, where p and q are the two generators of the Heisenberg algebra, while the relations (4.3) are motivated by the symmetric group relations.

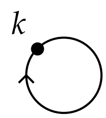
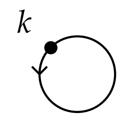
It is convenient to denote a right curl by a dot on a strand, and a sequence of d right curls by a dot with a d next to it:



The relations (4.3) allow us to identify elements of $\mathbb{C}[S_n]$ with linear combinations of diagrams with n upward oriented strands. For $x \in \mathbb{C}[S_n]$ our notation for such a linear combination of diagrams is a box with an x in it



Next set

$c_k \quad :=$  and $\tilde{c}_k \quad :=$  .

Theorem 9. *[10, Prop. 3] There are algebra isomorphisms*

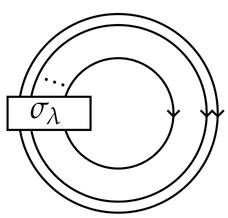
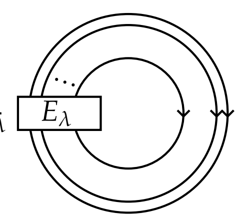
$$\mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \cong \mathbb{C}[c_0, c_1, \dots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \dots]. \tag{4.4}$$

Note that it follows from the relations in (4.2) that $\tilde{c}_0 = 1$ and $\tilde{c}_1 = 0$.

Lemma 1. *[10, Prop. 2] For $k > 0$,*

$$\tilde{c}_{k+1} = \sum_{i=0}^{k-1} \tilde{c}_i c_{k-1-i}. \tag{4.5}$$

Let E_λ be the Young idempotent associated with λ so that $\mathbb{C}[S_n]E_\lambda \cong L^\lambda$. Also let $\sigma_\lambda \in S_n$ be an element of cycle type λ and set

$\alpha_\lambda \quad :=$  , $\quad \tilde{E}_\lambda \quad := \dfrac{1}{\dim L^\lambda}$  .

Because the diagrams are closed, the local relations imply that all choices of σ_λ give the same element of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$, so α_λ is well-defined. We write $\alpha_k := \alpha_{(k)}$.

Proposition 10. *The elements $\alpha_1, \alpha_2, \dots$ are algebraically independent generators of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$.*

For each $n \geqslant 0$, Khovanov defines a functor $f_n^{\mathcal{H}'} : \mathcal{H}' \to \mathcal{S}'_n$, where $\mathcal{S}'_n$ is a bimodule category for symmetric groups whose objects are all right $\mathbb{C}[S_n]$-modules (see [10] for details). When restricted to $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$, $f_n^{\mathcal{H}'}$ can be interpreted as a surjective homomorphism into $Z(\mathbb{C}[S_n])$. Below we give the value of $f_n^{\mathcal{H}'}$ on c_k, $\tilde{c}_k$, and α_k in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$.

Lemma 2. *If $n \geqslant 1$, then*

1. $f_n^{\mathcal{H}'}(c_k) = \sum_{i=1}^{n} s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i$,
2. $f_n^{\mathcal{H}'}(\tilde{c}_k) = pr_n(J_{n+1}^k)$.
3. $f_n^{\mathcal{H}'}(\alpha_\mu) = \begin{cases} A_{\mu,n} & \text{if } |\mu| \leqslant n \\ 0 & \text{otherwise.} \end{cases}$

5 The isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow \Lambda^*$

In this section we establish the algebra isomorphism $\mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \cong \Lambda^*$. The proof is somewhat analogous to a proof of Ivanov and Kerov [5, Theorem 9.1].

For any $\lambda \in \mathcal{P}_n$, composing $f_n^{\mathcal{H}'}$ with the normalized character $\widetilde{\chi}^\lambda$ gives a map

$$(\widetilde{\chi}^\lambda \circ f_n^{\mathcal{H}'}) : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to \mathbb{C}$$

and allows us to define a homomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to \mathrm{Fun}(\mathcal{P}, \mathbb{C})$. For $x \in \mathrm{End}_{\mathcal{H}'}(\mathbf{1})$,

$$[\varphi(x)](\lambda) := (\widetilde{\chi}^\lambda \circ f_n^{\mathcal{H}'})(x).$$

Combining part 3 of Lemma 2 with equation (2.3) implies that for $\mu \in \mathcal{P}_k$

$$[\varphi(\alpha_\mu)](\lambda) = \begin{cases} \frac{(n\downarrow k)}{\dim L^\lambda}\chi^\lambda(\mu \cup 1^{n-k}) & \text{if } k \leqslant n \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Theorem 11. *The map φ induces an algebra isomorphism $\mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to \Lambda^* \subseteq \mathit{Fun}(\mathcal{P}, \mathbb{C})$ with $\alpha_\mu \overset{\varphi}{\mapsto} p_\mu^\#$.*

Proof. Let $\lambda \in \mathcal{P}_n$. φ is an algebra homomorphism because $f_n^{\mathcal{H}'}$ is a homomorphism from $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ to $Z(\mathbb{C}[S_n])$ and $\widetilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[S_n])$. By Proposition 5 and (5.1), α_μ maps to $p_\mu^\#$. Since the $\{p_k^\#\}_{k\geqslant 1}$ (respectively $\{\alpha_k\}_{k\geqslant 1}$) are algebraically independent generators of Λ^* (respectively $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$), φ must be an isomorphism. □

Corollary 12. *The isomorphism φ sends $\tilde{E}_\lambda \overset{\varphi}{\mapsto} s_\lambda^*$.*

Theorem 11 and Corollary 12 give graphical realizations of some important bases of Λ^*. Now we go the other way, and describe Khovanov's curl generators $\tilde{c}_k$ and c_k as elements of Λ^*. It is this description that makes an explicit connection between $\mathcal{H}'$ and the transition and co-transition measures of Kerov.

Theorem 13. *The isomorphism φ sends:*

1. $\tilde{c}_k \mapsto \widehat{m}_k \in \Lambda^*$,
2. $c_k \mapsto p_1^\# \check{m}_k = \widehat{b}_{k+2} \in \Lambda^*$.

Proof. This follows from Proposition 4 and Lemma 2. □

Remark 14. *Theorem 13 and Remark 8 together imply that the recursive relationships for $\{\widehat{m}_k\}$ and $\{\widehat{b}_k\}$ and the recursive relationships for $\{c_k\}$ and $\{\tilde{c}_k\}$ in Lemma 1 are both consequences of the well-known relationship between the elementary and homogeneous symmetric functions:*

$$\sum_{i=0}^{k}(-1)^i e_i h_{n-i} = 0.$$

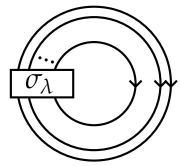
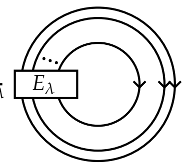
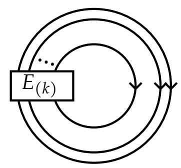
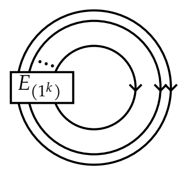
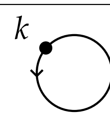
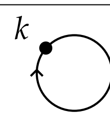
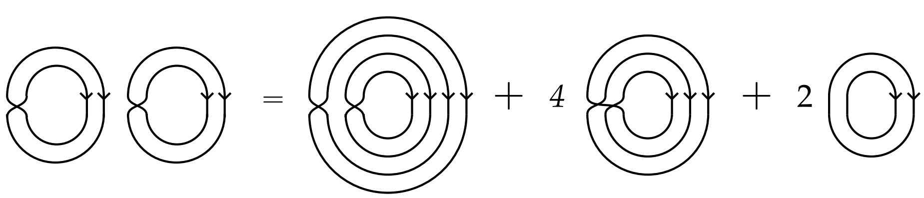
Λ^*	diagram in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$
$p^{\#}_{\lambda}$	
s^*_{λ}	$\frac{1}{\dim L^{\lambda}}$ 
h^*_k	
e^*_k	
$\widehat{m}_k$	
$\widehat{b}_{k+2} = p^{\#}_1 \check{m}_k$	

Table 1: A dictionary between Λ^* and diagrams in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$.

Example 15. *In* Λ^* *we have* $p^{\#}_{(2)}p^{\#}_{(2)} = p^{\#}_{(2,2)} + 4p^{\#}_{(3)} + 2p^{\#}_{(1,1)}$. *In* $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ *this appears as*



and can be computed independently via the local relations.

In [10], Khovanov introduced three involutive autoequivalences on $\mathcal{H}'$. Only one of these, which we denote as ξ, acts non-trivially on $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ where it gives an involutive algebra automorphism. For diagram $D \in \mathrm{End}_{\mathcal{H}'}(\mathbf{1})$, we have $\xi(D) := (-1)^{c(D)}D$ where

$c(D)$ is the total number of dots and crossings in the diagram. In Section 4 of [13], Okounkov and Olshanski identified an involutive algebra automorphism $I : \Lambda^* \to \Lambda^*$ such that for $f \in \Lambda^*$ and $\lambda \in \mathcal{P}$, $[I(f)](\lambda) = f(\lambda')$ where λ' is the conjugate partition to λ.

Proposition 16. *The involution* ξ *on* $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ *coincides with the involution* I *on* Λ^*.

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